

Uniform Triangles with Equality Constraints

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ABSTRACT. The equality constraint $a + b + c = 1$ for random triangle sides corresponds to breaking a stick in two places. An analog $a^2 + b^2 + c^2 = 1$ has a remarkable feature: the bivariate density for angles coincides with that for 3D Gaussian triangles. Interesting complications also arise for $a + b = 1$ and for $a^2 + b^2 = 1$, with the understanding that the angle γ opposite side c is Uniform $[0, \pi]$. Closed-form expressions for several side moments remain open.

There is a natural method for generating triangles of unit perimeter: break a stick of length 1 in two places at random, with the condition that triangle inequalities are satisfied. We denote triangle sides by a, b, c and opposite angles by α, β, γ . Pointers to the literature are found in [1, 2]; the bivariate density for two arbitrary sides

$$\begin{cases} 8 & \text{if } 0 < x < 1/2, 0 < y < 1/2 \text{ and } x + y > 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

is well-known. Proof of the bivariate density for two arbitrary angles

$$\begin{cases} 8 \frac{\sin(x) \sin(y) \sin(x+y)}{(\sin(x) + \sin(y) + \sin(x+y))^3} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x + y < \pi, \\ 0 & \text{otherwise} \end{cases}$$

will be given in Section 1. The latter is a new result, as far as is known, although it bears resemblance to formulas in [3]. In particular, the probability that such a triangle is obtuse is $9 - 12 \ln(2) \approx 0.682$.

Let the lengths of the three pieces (from breaking the stick) instead be a^2, b^2, c^2 . Inspiration for this example came from Edelman & Strang [4]. We will prove in Section 2 that the bivariate side density is

$$\begin{cases} \frac{24\sqrt{3}}{\pi} x y & \text{if } |x - y| < \sqrt{1 - x^2 - y^2} < x + y, \\ 0 & \text{otherwise} \end{cases}$$

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and the bivariate angle density is

$$\begin{cases} \frac{24\sqrt{3}}{\pi} \frac{\sin(x)^2 \sin(y)^2 \sin(x+y)^2}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2)^3} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x+y < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

The probability that such a triangle is obtuse is $1 - 3\sqrt{3}/(4\pi) \approx 0.586$. It is remarkable that angles here are distributed identically to angles for Gaussian triangles in three-dimensional space [1]. There is no *a priori* reason to expect such a coincidence. One class of triangles arises synthetically (from breaking a stick: the ambient space doesn't matter) while the other class arises analytically (via a sampling of vertices, *i.e.*, point coordinates in \mathbb{R}^3 : the ambient space matters).

Let us instead break the stick in just one place at random, giving sides a and b . Generate independently and uniformly an angle γ from the interval $[0, \pi]$. The remaining side c and angles α, β are computed via the Law of Cosines. We will prove in Section 3 that the bivariate side density for $a = x, c = y$ is

$$\begin{cases} \frac{2}{\pi} \frac{y}{\sqrt{1-y^2} \sqrt{4x(1-x) - (1-y^2)}} & \text{if } |2x-1| < y < 1, \\ 0 & \text{otherwise} \end{cases}$$

and the bivariate angle density for angles α, β is

$$\begin{cases} \frac{1}{\pi} \frac{\sin(x+y)}{(\sin(x) + \sin(y))^2} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x+y < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the side density is complicated while the angle density is simple. The probability that such a triangle is obtuse is $3/2 - 2/\pi \approx 0.863$.

Let the lengths of the two pieces (from breaking the stick) instead be a^2, b^2 . The angle γ is exactly as before. We will prove in Section 4 that the bivariate side density for $a = x, c = y$ is

$$\begin{cases} \frac{4}{\pi} \frac{xy}{\sqrt{4x^2(1-x^2) - (1-y^2)^2}} & \text{if } |x - \sqrt{1-x^2}| < y < x + \sqrt{1-x^2}, \\ 0 & \text{otherwise} \end{cases}$$

and the bivariate angle density for angles α, β is

$$\begin{cases} \frac{2}{\pi} \frac{\sin(x) \sin(y) \sin(x+y)}{(\sin(x)^2 + \sin(y)^2)^2} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x+y < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

The probability that such a triangle is obtuse is $3/2 - 1/\sqrt{2} \approx 0.793$. Another remarkable coincidence occurs here: angles are distributed identically to angles for pinned Gaussian triangles in two-dimensional space [3].

For variety's sake, select a point (a, b) uniformly on the positive quarter circle of unit radius, center at $(0, 0)$. Although $a^2 + b^2 = 1$ here as well, sampling from a circle is different from breaking a stick/extracting square roots. The angle γ is exactly as before. Inspiration for this example came from Portnoy [5]. We will prove in Section 5 that the bivariate side density for $a = x$, $c = y$ is

$$\begin{cases} \frac{4}{\pi^2} \frac{y}{\sqrt{1-x^2} \sqrt{4x^2(1-x^2) - (1-y^2)^2}} & \text{if } |x - \sqrt{1-x^2}| < y < x + \sqrt{1-x^2}, \\ 0 & \text{otherwise} \end{cases}$$

and the bivariate angle density for angles α , β is

$$\begin{cases} \frac{2}{\pi^2} \frac{\sin(x+y)}{\sin(x)^2 + \sin(y)^2} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x+y < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Again the side density is complicated while the angle density is simple. The probability that such a triangle is obtuse is $1 - (2/\pi^2) \ln(1 + \sqrt{2})^2 \approx 0.842$.

Finally, select a point (a, b, c) uniformly on the positive one-eighth sphere of unit radius, center at $(0, 0, 0)$. Although $a^2 + b^2 + c^2 = 1$ here like Section 2, sampling from a sphere is different. We will prove in Section 6 that the bivariate side density is

$$\begin{cases} \frac{C}{\sqrt{x^2 + y^2} \sqrt{1 - x^2 - y^2}} & \text{if } |x - y| < \sqrt{1 - x^2 - y^2} < x + y, \\ 0 & \text{otherwise} \end{cases}$$

and the bivariate angle density is

$$\begin{cases} \frac{C \sin(x) \sin(y) \sin(x+y)}{(\sin(x)^2 + \sin(y)^2 + \sin(x+y)^2) \sqrt{\sin(x)^2 + \sin(y)^2}} & \text{if } 0 < x < \pi, 0 < y < \pi \text{ and } x+y < \pi, \\ 0 & \text{otherwise} \end{cases}$$

where C is a known constant (written in terms of dilogarithm function values). The probability that such a triangle is obtuse is approximately 0.659. This is the first of several quantities appearing here for which explicit formulation is unavailable. One quantity, given as approximately 0.958 in Section 5, is especially important to understand more fully. Insight and help toward unraveling these constants would be appreciated.

1. CONSTRAINT $a + b + c = 1$

If we break a line segment in two places at random, the three pieces can be configured as a triangle with probability $1/4$. Reason: the subdomain $\{(x, y) : 0 < x < 1/2, 0 < y < 1/2, x + y > 1/2\}$ occupies one-fourth the area of domain $\{(x, y) : 0 < x < 1, 0 < y < 1, x + y < 1\}$, and the triangle inequalities

$$1 - a - b = c < a + b, \quad a < b + c = 1 - a, \quad b < a + c = 1 - b$$

become $a + b > 1/2$, $a < 1/2$ and $b < 1/2$. The Law of Sines gives

$$b \sin(\alpha) - a \sin(\beta) = 0$$

and the Law of Cosines gives

$$\begin{aligned} b \cos(\alpha) + a \cos(\beta) &= \frac{-a^2 + b^2 + (1 - a - b)^2}{2(1 - a - b)} + \frac{a^2 - b^2 + (1 - a - b)^2}{2(1 - a - b)} \\ &= 1 - a - b. \end{aligned}$$

Solving for a, b yields

$$a = \frac{\sin(\alpha)}{\sin(\alpha) + \sin(\beta) + \sin(\alpha + \beta)}, \quad b = \frac{\sin(\beta)}{\sin(\alpha) + \sin(\beta) + \sin(\alpha + \beta)}$$

and thus

$$1 - a - b = \frac{\sin(\alpha + \beta)}{\sin(\alpha) + \sin(\beta) + \sin(\alpha + \beta)}.$$

Now, the map $(a, b) \mapsto (\alpha, \beta)$ defined via the Law of Cosines has Jacobian determinant

$$|J| = \frac{1}{ab(1 - a - b)}$$

hence the desired bivariate density for angles is

$$\frac{8}{|J|} = 8 \frac{\sin(\alpha) \sin(\beta) \sin(\alpha + \beta)}{(\sin(\alpha) + \sin(\beta) + \sin(\alpha + \beta))^3}.$$

We have univariate densities

$$\begin{aligned} &\begin{cases} 8a & \text{if } 0 < a < 1/2, \\ 0 & \text{otherwise;} \end{cases} \\ &\begin{cases} -8 \frac{(3 - \cos(\alpha)) \sin(\alpha)}{(1 + \cos(\alpha))^3} \ln \left(\sin \left(\frac{\alpha}{2} \right) \right) - 8 \frac{\sin(\alpha)}{(1 + \cos(\alpha))^2} & \text{if } 0 < \alpha < \pi, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and moments

$$\begin{aligned} E(a) &= 1/3, & E(a^2) &= 1/8, & E(ab) &= 5/48; \\ E(\alpha) &= \pi/3, & E(\alpha^2) &= 8/3 - \pi^2/9, & E(\alpha\beta) &= -4/3 + 2\pi^2/9. \end{aligned}$$

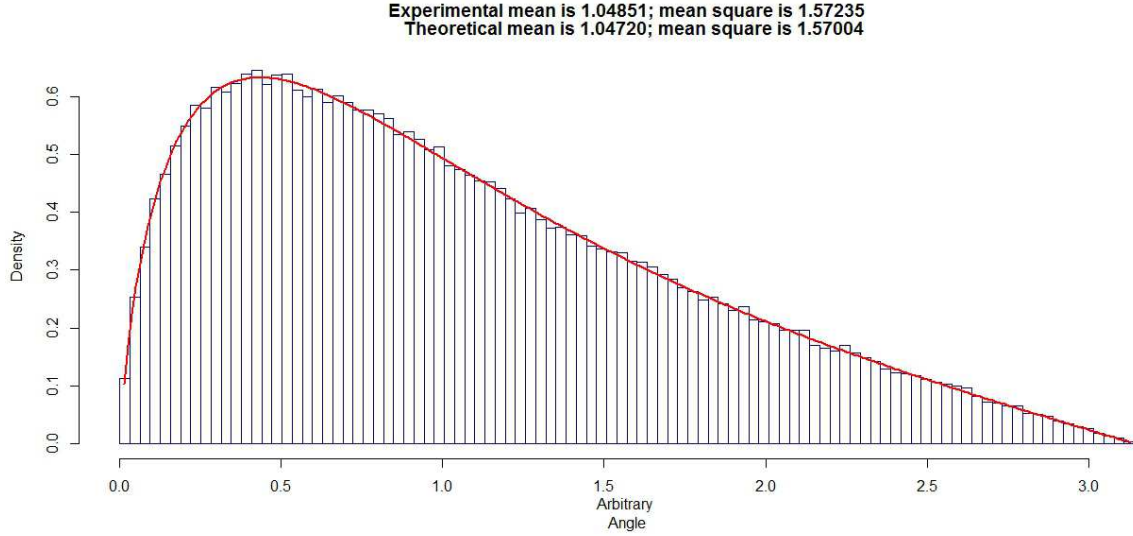


Figure 1: Density function for arbitrary angle in Section 1.

2. CONSTRAINT $a^2 + b^2 + c^2 = 1$

Here a^2, b^2 are uniform, not a, b . The triangle inequalities

$$\sqrt{1 - a^2 - b^2} = c < a + b,$$

$$a < b + c = b + \sqrt{1 - a^2 - b^2},$$

$$b < a + c = a + \sqrt{1 - a^2 - b^2}$$

are simultaneously met with probability $\sqrt{3}\pi/9 \approx 0.604$ [4] because the subdomain

$$\left\{ (x, y) : |x - y| < \sqrt{1 - x^2 - y^2} < x + y \right\}$$

has area $\sqrt{3}\pi/18$ whereas the domain $\{(x, y) : 0 < x < 1, 0 < y < 1, x + y < 1\}$ has area $1/2$. The map $(u, v) \mapsto (\sqrt{u}, \sqrt{v})$ has Jacobian determinant $1/(4\sqrt{uv})$, providing the form ab of the bivariate side density. As before, the Law of Sines gives

$$b \sin(\alpha) - a \sin(\beta) = 0$$

and the Law of Cosines gives

$$\begin{aligned} b \cos(\alpha) + a \cos(\beta) &= \frac{-a^2 + b^2 + (1 - a^2 - b^2)}{2\sqrt{1 - a^2 - b^2}} + \frac{a^2 - b^2 + (1 - a^2 - b^2)}{2\sqrt{1 - a^2 - b^2}} \\ &= \sqrt{1 - a^2 - b^2}. \end{aligned}$$

Solving for a, b yields

$$a = \frac{\sin(\alpha)}{\sqrt{\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2}}, \quad b = \frac{\sin(\beta)}{\sqrt{\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2}}$$

and thus

$$\sqrt{1 - a^2 - b^2} = \frac{\sin(\alpha + \beta)}{\sqrt{\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2}}.$$

Now, the map $(a, b) \mapsto (\alpha, \beta)$ defined via the Law of Cosines has Jacobian determinant

$$|J| = \frac{1}{ab(1 - a^2 - b^2)}$$

hence the desired bivariate density for angles is

$$\frac{24\sqrt{3}}{\pi} \frac{ab}{|J|} = \frac{24\sqrt{3}}{\pi} \frac{\sin(\alpha)^2 \sin(\beta)^2 \sin(\alpha + \beta)^2}{(\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha + \beta)^2)^3}.$$

We have univariate densities

$$\begin{cases} \frac{12\sqrt{3}}{\pi} a^2 \sqrt{2 - 3a^2} & \text{if } 0 < a < \frac{\sqrt{6}}{3}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\begin{cases} \frac{6\sqrt{3}}{\pi} \frac{(2 + \cos(\alpha)^2) \sin(\alpha)}{(4 - \cos(\alpha)^2)^{5/2}} \left(\frac{\pi}{2} + \arcsin\left(\frac{\cos(\alpha)}{2}\right) \right) + \frac{9\sqrt{3}}{\pi} \frac{\cos(\alpha) \sin(\alpha)}{(4 - \cos(\alpha)^2)^2} & \text{if } 0 < \alpha < \pi, \\ 0 & \text{otherwise} \end{cases}$$

and moments

$$E(a) = \frac{32\sqrt{6}}{45\pi}, \quad E(a^2) = \frac{1}{3}, \quad E(ab) = \frac{9 + \sqrt{3}\pi}{9\sqrt{3}\pi};$$

$$E(\alpha) = \frac{\pi}{3}, \quad E(\alpha^2) = \frac{1}{3} (\pi - \sqrt{3}) \pi, \quad E(\alpha\beta) = \frac{\sqrt{3}\pi}{6}.$$

We are familiar with the angle density [3]; the side density, however, is new.

3. CONSTRAINT $a + b = 1$ AND UNIFORM γ

The map $(a, \gamma) \mapsto (a, c)$ defined by

$$c^2 = a^2 + (1 - a)^2 - 2a(1 - a) \cos(\gamma)$$

has Jacobian determinant

$$\begin{aligned}
|J| &= \frac{\partial c}{\partial \gamma} = \frac{1}{2c} \cdot 2a(1-a) \sin(\gamma) \\
&= \frac{a(1-a)}{c} \sqrt{1 - \cos(\gamma)^2} \\
&= \frac{a(1-a)}{c} \sqrt{1 - \left[\frac{a^2 + (1-a)^2 - c^2}{2a(1-a)} \right]^2} \\
&= \frac{a(1-a)}{c} \sqrt{\frac{4a^2(1-a)^2 - [a^2 + (1-a)^2 - c^2]^2}{4a^2(1-a)^2}} \\
&= \frac{1}{2c} \sqrt{4a^2(1-a)^2 - [2a^2 - 2a + 1 - c^2]^2} \\
&= \frac{1}{2c} \sqrt{4a^2(1-a)^2 - [-2a(1-a) + (1-c^2)]^2} \\
&= \frac{1}{2c} \sqrt{4a(1-a)(1-c^2) - (1-c^2)^2} \\
&= \frac{1}{2c} \sqrt{1-c^2} \sqrt{4a(1-a) - (1-c^2)};
\end{aligned}$$

therefore the bivariate side density is

$$\frac{1}{\pi} \frac{1}{|J|} = \frac{2}{\pi} \frac{c}{\sqrt{1-c^2} \sqrt{4a(1-a) - (1-c^2)}}.$$

The Law of Sines gives

$$c \sin(\alpha) - a \sin(\alpha + \beta) = 0$$

and the Law of Cosines gives

$$\begin{aligned}
(1-a) \cos(\alpha) + a \cos(\beta) &= b \cos(\alpha) + a \cos(\beta) \\
&= \frac{-a^2 + b^2 + c^2}{2c} + \frac{a^2 - b^2 + c^2}{2c} = c.
\end{aligned}$$

Solving for a, c yields

$$a = \frac{\sin(\alpha)}{\sin(\alpha) + \sin(\beta)}, \quad c = \frac{\sin(\alpha + \beta)}{\sin(\alpha) + \sin(\beta)}$$

and thus

$$b = 1 - a = \frac{\sin(\beta)}{\sin(\alpha) + \sin(\beta)}.$$

Now, the map $(a, c) \mapsto (\alpha, \beta)$ defined via the Law of Cosines has Jacobian determinant

$$|I| = \frac{1}{a b c}$$

hence the desired bivariate density for angles is

$$\begin{aligned} \frac{2}{\pi} \frac{c}{\sqrt{1-c^2} \sqrt{4a(1-a) - (1-c^2)}} \frac{1}{|I|} &= \frac{1}{\pi} \frac{\sin(\alpha) + \sin(\beta)}{\sin(\alpha) \sin(\beta)} \frac{\sin(\alpha) \sin(\beta) \sin(\alpha + \beta)}{(\sin(\alpha) + \sin(\beta))^3} \\ &= \frac{1}{\pi} \frac{\sin(\alpha + \beta)}{(\sin(\alpha) + \sin(\beta))^2}. \end{aligned}$$

We have univariate densities

$$\begin{aligned} &\begin{cases} 1 & \text{if } 0 < a < 1, \\ 0 & \text{otherwise;} \end{cases} \quad \begin{cases} \frac{c}{\sqrt{1-c^2}} & \text{if } 0 < c < 1, \\ 0 & \text{otherwise;} \end{cases} \\ &\begin{cases} -\frac{1}{\pi} \frac{1}{\cos(\alpha)^2} \left[2 \ln \left(\sin \left(\frac{\alpha}{2} \right) \right) + \ln(2) \right] - \frac{1}{\pi} \frac{1}{\cos(\alpha)} & \text{if } 0 < \alpha < \pi, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and moments

$$\begin{aligned} E(a) &= \frac{1}{2}, & E(a^2) &= \frac{1}{3}, & E(c) &= \frac{\pi}{4}, & E(c^2) &= \frac{2}{3}, & E(ac) &= \frac{\pi}{8}; \\ E(\alpha) &= \frac{\pi}{4}, & E(\alpha^2) &= 1.3029200473..., & E(\alpha\beta) &= 0.3420140195.... \end{aligned}$$

4. CONSTRAINT $a^2 + b^2 = 1$ AND UNIFORM γ

The map $(a, \gamma) \mapsto (a, c)$ defined by

$$c^2 = a^2 + (1 - a^2) - 2a\sqrt{1 - a^2} \cos(\gamma)$$

has Jacobian determinant

$$\begin{aligned} |J| &= \frac{\partial c}{\partial \gamma} = \frac{1}{2c} \cdot 2a\sqrt{1 - a^2} \sin(\gamma) \\ &= \frac{a\sqrt{1 - a^2}}{c} \sqrt{1 - \cos(\gamma)^2} \\ &= \frac{a\sqrt{1 - a^2}}{c} \sqrt{1 - \left[\frac{a^2 + (1 - a^2) - c^2}{2a\sqrt{1 - a^2}} \right]^2} \\ &= \frac{a\sqrt{1 - a^2}}{c} \sqrt{\frac{4a^2(1 - a^2) - (1 - c^2)^2}{4a^2(1 - a^2)}} \\ &= \frac{1}{2c} \sqrt{4a^2(1 - a^2) - (1 - c^2)^2}; \end{aligned}$$

therefore the bivariate side density is

$$\frac{1}{\pi} \frac{4a}{|J|} = \frac{4}{\pi} \frac{ac}{\sqrt{4a^2(1-a^2) - (1-c^2)^2}}.$$

The additional factor a comes because $u \mapsto \sqrt{u}$ has derivative $1/(2\sqrt{u})$. As before, the Law of Sines gives

$$c \sin(\alpha) - a \sin(\alpha + \beta) = 0$$

and the Law of Cosines gives

$$\begin{aligned} \sqrt{1-a^2} \cos(\alpha) + a \cos(\beta) &= b \cos(\alpha) + a \cos(\beta) \\ &= \frac{-a^2 + b^2 + c^2}{2c} + \frac{a^2 - b^2 + c^2}{2c} = c. \end{aligned}$$

Solving for a, c yields

$$a = \frac{\sin(\alpha)}{\sqrt{\sin(\alpha)^2 + \sin(\beta)^2}}, \quad c = \frac{\sin(\alpha + \beta)}{\sqrt{\sin(\alpha)^2 + \sin(\beta)^2}}$$

and thus

$$b^2 = 1 - a^2 = \frac{\sin(\beta)^2}{\sin(\alpha)^2 + \sin(\beta)^2}.$$

Now, the map $(a, c) \mapsto (\alpha, \beta)$ defined via the Law of Cosines has Jacobian determinant

$$|I| = \frac{1}{ab^2c}$$

hence the desired bivariate density for angles is

$$\begin{aligned} \frac{4}{\pi} \frac{ac}{\sqrt{4a^2(1-a^2) - (1-c^2)^2}} \frac{1}{|I|} &= \frac{2}{\pi} \frac{1}{\sin(\beta)} \frac{\sin(\alpha) \sin(\beta)^2 \sin(\alpha + \beta)}{(\sin(\alpha)^2 + \sin(\beta)^2)^2} \\ &= \frac{2 \sin(\alpha) \sin(\beta) \sin(\alpha + \beta)}{\pi (\sin(\alpha)^2 + \sin(\beta)^2)^2}. \end{aligned}$$

We have univariate densities

$$\begin{aligned} &\begin{cases} 2a & \text{if } 0 < a < 1, \\ 0 & \text{otherwise;} \end{cases} \quad \begin{cases} c & \text{if } 0 < c < \sqrt{2}, \\ 0 & \text{otherwise;} \end{cases} \\ &\begin{cases} \frac{1}{\pi} \frac{\cos(\alpha)}{(2 - \cos(\alpha)^2)^{3/2}} \left(\frac{\pi}{2} + \arcsin \left(\frac{\cos(\alpha)}{\sqrt{2}} \right) \right) + \frac{1}{\pi} \frac{1}{2 - \cos(\alpha)^2} & \text{if } 0 < \alpha < \pi, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and moments

$$E(a) = \frac{2}{3}, \quad E(a^2) = \frac{1}{2}, \quad E(c) = \frac{2\sqrt{2}}{3}, \quad E(c^2) = 1,$$

$$E(ac) = \frac{\sqrt{2}}{\pi} \int_0^{\sqrt{2}} t^2 \sqrt{1 + t\sqrt{2-t^2}} E\left(\sqrt{\frac{2t\sqrt{2-t^2}}{1+t\sqrt{2-t^2}}}\right) dt = 0.6272922529\dots$$

where

$$E(\xi) = \int_0^{\pi/2} \sqrt{1 - \xi^2 \sin^2(\theta)} d\theta = \int_0^1 \sqrt{\frac{1 - \xi^2 t^2}{1 - t^2}} dt$$

is the complete elliptic integral of the second kind;

$$E(\alpha) = \frac{\pi}{4}, \quad E(\alpha^2) = \frac{5}{48}\pi^2 + \frac{1}{4}\ln(2)^2, \quad E(\alpha\beta) = \frac{1}{16}\pi^2 - \frac{1}{4}\ln(2)^2.$$

We are familiar with the angle density [3]; the side density, however, is new.

5. CONSTRAINT $a^2 + b^2 = 1$ REVISITED

Let θ be uniformly distributed on the interval $[0, \pi/2]$. The map $(\theta, \gamma) \mapsto (a, c)$ defined by

$$\begin{cases} a = \cos(\theta), \\ c^2 = a^2 + (1 - a^2) - 2a\sqrt{1 - a^2} \cos(\gamma) \end{cases}$$

has Jacobian determinant

$$\begin{aligned} |J| &= \frac{\partial a}{\partial \theta} \frac{\partial c}{\partial \gamma} = \sin(\theta) \cdot \frac{1}{2c} \cdot 2a\sqrt{1 - a^2} \sin(\gamma) \\ &= \frac{a(1 - a^2)}{c} \sqrt{1 - \cos(\gamma)^2} \\ &= \frac{a(1 - a^2)}{c} \sqrt{1 - \left[\frac{a^2 + (1 - a^2) - c^2}{2a\sqrt{1 - a^2}} \right]^2} \\ &= \frac{a(1 - a^2)}{c} \sqrt{\frac{4a^2(1 - a^2) - (1 - c^2)^2}{4a^2(1 - a^2)}} \\ &= \frac{1}{2c} \sqrt{1 - a^2} \sqrt{4a^2(1 - a^2) - (1 - c^2)^2}; \end{aligned}$$

therefore the bivariate side density is

$$\frac{2}{\pi^2} \frac{1}{|J|} = \frac{4}{\pi^2} \frac{c}{\sqrt{1 - a^2} \sqrt{4a^2(1 - a^2) - (1 - c^2)^2}}.$$

The same expressions for $a, b, c, |I|$ apply as in Section 4, hence the desired bivariate density for angles is

$$\begin{aligned} \frac{4}{\pi^2} \frac{c}{\sqrt{1-a^2} \sqrt{4a^2(1-a^2) - (1-c^2)^2}} \frac{1}{|I|} &= \frac{2}{\pi^2} \frac{\sin(\alpha)^2 + \sin(\beta)^2}{\sin(\alpha) \sin(\beta)^2} \frac{\sin(\alpha) \sin(\beta)^2 \sin(\alpha + \beta)}{(\sin(\alpha)^2 + \sin(\beta)^2)^2} \\ &= \frac{2}{\pi^2} \frac{\sin(\alpha + \beta)}{\sin(\alpha)^2 + \sin(\beta)^2}. \end{aligned}$$

We have univariate densities

$$\begin{cases} \frac{2}{\pi} \frac{1}{\sqrt{1-a^2}} & \text{if } 0 < a < 1, \\ 0 & \text{otherwise;} \end{cases} \quad \begin{cases} \frac{4c}{\pi^2} K(c\sqrt{2-c^2}) & \text{if } 0 < c < \sqrt{2}, \\ 0 & \text{otherwise} \end{cases}$$

where

$$K(\xi) = \int_0^{\pi/2} \frac{1}{\sqrt{1-\xi^2 \sin^2(\theta)}} d\theta = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\xi^2 t^2)}} dt$$

is the complete elliptic integral of the first kind;

$$\begin{cases} \frac{1}{2\pi} + \frac{1}{\pi^2} \frac{\cos(\alpha)}{\sqrt{2-\cos(\alpha)^2}} \ln \left(\frac{2-\cos(\alpha) + \sqrt{2-\cos(\alpha)^2}}{2-\cos(\alpha) - \sqrt{2-\cos(\alpha)^2}} \right) & \text{if } 0 < \alpha < \pi, \\ 0 & \text{otherwise} \end{cases}$$

and moments

$$\begin{aligned} E(a) &= \frac{2}{\pi}, \quad E(a^2) = \frac{1}{2}, \quad E(c) = 0.9580913986..., \quad E(c^2) = 1, \\ E(ac) &= \frac{2\sqrt{2}}{\pi^2} \int_0^{\sqrt{2}} \frac{t^2}{\sqrt{1+t\sqrt{2-t^2}}} K\left(\sqrt{\frac{2t\sqrt{2-t^2}}{1+t\sqrt{2-t^2}}}\right) dt = 0.6080033617..., \\ E(\alpha) &= \frac{\pi}{4}, \quad E(\alpha^2) = 1.2565739217..., \quad E(\alpha\beta) = 0.3883601451.... \end{aligned}$$

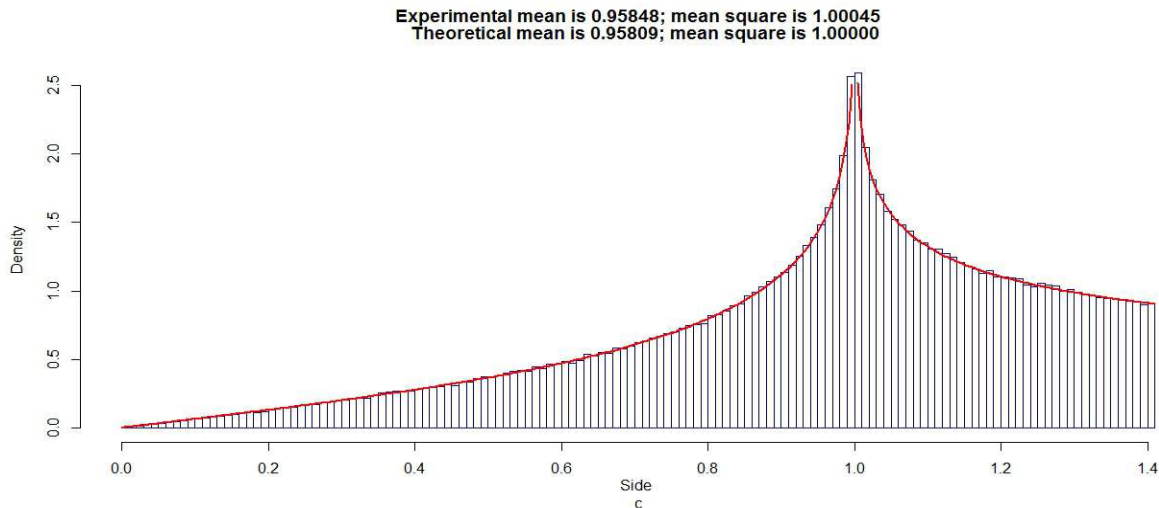
Evaluating the mean of side c in closed-form remains tantalizingly open. In addition to its definition:

$$E(c) = \frac{4}{\pi^2} \int_0^{\sqrt{2}} t^2 K(t\sqrt{2-t^2}) dt$$

we have the following representation:

$$E(c) = \frac{2}{\pi^2} \int_0^1 s \frac{\sqrt{1-\sqrt{1-s^2}} + \sqrt{1+\sqrt{1-s^2}}}{\sqrt{1-s^2}} K(s) ds$$

which unfortunately does not appear in [6].

Figure 2: Density function for side c in Section 5.

6. CONSTRAINT $a^2 + b^2 + c^2 = 1$ REVISITED

Let φ, ψ be independently and uniformly distributed on the interval $[0, \pi/2]$. The map $(\varphi, \psi) \mapsto (a, b)$ defined by

$$\begin{cases} a = \sin(\varphi) \cos(\psi), \\ b = \sin(\varphi) \sin(\psi) \end{cases}$$

has Jacobian determinant

$$|J| = \cos(\varphi) \sin(\varphi) = \sqrt{a^2 + b^2} \sqrt{1 - a^2 - b^2}$$

therefore the bivariate side density is

$$\frac{C}{|J|} = \frac{C}{\sqrt{a^2 + b^2} \sqrt{1 - a^2 - b^2}}$$

and the normalizing constant satisfies

$$\begin{aligned}
 \frac{1}{C} &= 2 \int_0^1 \frac{\arctan(x\sqrt{1+x^2})}{1+x^2} dx \\
 &= \frac{\pi}{2} \arctan(\sqrt{2}) - 2 \int_0^1 \frac{1+2x^2}{\sqrt{1+x^2}(1+x^2+x^4)} dx \\
 &= -\frac{\pi^2}{24} - \frac{1}{2} i \pi \ln(2-\sqrt{3}) - \text{Li}_2\left(\sqrt{2-\sqrt{3}}\right) + \\
 &\quad \text{Li}_2\left(-\sqrt{2-\sqrt{3}}\right) - \text{Li}_2\left(-\sqrt{2+\sqrt{3}}\right) + \text{Li}_2\left(\sqrt{2+\sqrt{3}}\right) + \\
 &\quad \text{Li}_2\left(\frac{1-i}{1+\sqrt{3}}\right) + \text{Li}_2\left(\frac{1+i}{1+\sqrt{3}}\right) + \text{Li}_2\left(\frac{1+\sqrt{3}}{-1+i}\right) + \text{Li}_2\left(\frac{1+\sqrt{3}}{-1-i}\right) \\
 &= 0.6947951075\dots
 \end{aligned}$$

where

$$\text{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = - \int_0^x \frac{\ln(1-t)}{t} dt$$

is the dilogarithm function. The same expressions for a , b , c , $|I|$ apply as in Section 2, hence the desired bivariate density for angles is

$$\frac{C a b \sqrt{1-a^2-b^2}}{\sqrt{a^2+b^2}} = \frac{C \sin(\alpha) \sin(\beta) \sin(\alpha+\beta)}{(\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha+\beta)^2) \sqrt{\sin(\alpha)^2 + \sin(\beta)^2}}.$$

Let

$$\omega(a) = \arcsin \left(\frac{a + \sqrt{2-3a^2}}{\sqrt{1-a^2} \sqrt{4a^2 + (a + \sqrt{2-3a^2})^2}} \right)$$

then the univariate side density is

$$\begin{cases} -C F(\omega(-a), \sqrt{1-a^2}) + C F(\omega(a), \sqrt{1-a^2}) & \text{if } 0 < a < \sqrt{1/2}, \\ -C F(-\omega(-a), \sqrt{1-a^2}) + C F(\omega(a), \sqrt{1-a^2}) & \text{if } \sqrt{1/2} < a < \sqrt{2/3}, \\ 0 & \text{otherwise} \end{cases}$$

where

$$F(\omega, \xi) = \int_0^\omega \frac{1}{\sqrt{1-\xi^2 \sin(\theta)^2}} d\theta = \int_0^{\sin(\omega)} \frac{1}{\sqrt{(1-t^2)(1-\xi^2 t^2)}} dt$$

is the incomplete elliptic integral of the first kind. We have not attempted to evaluate the univariate angle density; numerical calculations lead to moments:

$$E(a) = 0.5361308550..., \quad E(a^2) = 0.3209403207..., \quad E(ab) = 0.2707436816...,$$

$$E(\alpha) = 1.0018939715..., \quad E(\alpha^2) = 1.4360872743..., \quad E(\alpha\beta) = 0.8093206054...$$

none of which are immediately recognizable.

For the sake of thoroughness (but without proof), the pieces a , b , c can be configured as a triangle with probability

$$\begin{aligned} \Delta &= \frac{8}{\pi^2} \int_{\lambda}^{\pi/4} \left[\frac{\pi}{4} - 2 \arctan \left(\frac{\sin(\varphi) - \sqrt{-\cos(\varphi)^2 + 2\sin(\varphi)^2}}{\cos(\varphi) + \sin(\varphi)} \right) \right] d\varphi + \\ &\quad \frac{8}{\pi^2} \int_{\pi/4}^{\pi/2} \left[\frac{\pi}{4} - 2 \arctan \left(\frac{-\sin(\varphi) + \sqrt{-\cos(\varphi)^2 + 2\sin(\varphi)^2}}{\cos(\varphi) + \sin(\varphi)} \right) \right] d\varphi \\ &= 0.2815898507... \end{aligned}$$

where $\lambda = \arccos(\sqrt{2/3})$ and, conditional on this, the obtuseness probability is

$$1 - \frac{8}{\Delta\pi^2} \int_{\pi/4}^{\pi/2} \left[\frac{\pi}{4} - 2 \arccos \left(\frac{1}{\sqrt{2}\sin(\varphi)} \right) \right] d\varphi = 0.6597451305....$$

7. INTEGRATION DETAILS

Starting with the bivariate density for sides a , c in Section 5, let $x = 2a^2 - 1$, then

$$dx = 4a da, \quad 1 - a^2 = 1 - \left(\frac{1+x}{2} \right) = \frac{1-x}{2}$$

hence

$$\frac{\sqrt{2}}{4\sqrt{1+x}} dx = da, \quad 4a^2(1-a^2) = 2(1+x) \left(\frac{1-x}{2} \right) = 1-x^2$$

and thus the density becomes

$$\frac{4}{\pi^2} \frac{\sqrt{2}}{\sqrt{1-x}} \frac{c}{\sqrt{(1-x^2) - (1-c^2)^2}} \frac{\sqrt{2}}{4\sqrt{1+x}} = \frac{2}{\pi^2} \frac{c}{\sqrt{1-x^2} \sqrt{[1 - (1-c^2)^2] - x^2}}.$$

Integrating with respect to x , formula 3.152.7 in [7] can be applied to obtain the univariate density for c . To compute $E(ac)$, examine instead

$$\frac{2}{\pi^2} \frac{ac^2}{\sqrt{1-x^2}\sqrt{[1-(1-c^2)^2]-x^2}} = \frac{\sqrt{2}}{\pi^2} \frac{c^2}{\sqrt{1-x}\sqrt{[1-(1-c^2)^2]-x^2}}$$

and apply formula 3.131.3 in [7]. The analogous expression for $E(ac)$ in Section 4:

$$\frac{4}{\pi} \frac{a^2c^2}{\sqrt{(1-x^2)-(1-c^2)^2}} \frac{\sqrt{2}}{4\sqrt{1+x}} = \frac{1}{\sqrt{2}\pi} \frac{c^2\sqrt{1+x}}{\sqrt{[1-(1-c^2)^2]-x^2}}$$

can be integrated via formula 3.141.17 in [7].

Starting with the bivariate density for sides a, b in Section 6, let

$$x = \sqrt{a^2 + b^2}, \quad y = a/b$$

then

$$|J| = \frac{1+y^2}{x}, \quad a = \frac{xy}{\sqrt{1+y^2}}, \quad b = \frac{x}{\sqrt{1+y^2}}$$

and the density becomes separable:

$$\frac{1}{\sqrt{1-x^2}(1+y^2)}$$

although the region of integration initially appears unmanageable. The inequalities

$$\frac{x|y-1|}{\sqrt{1+y^2}} < \sqrt{1-x^2} < \frac{x(y+1)}{\sqrt{1+y^2}}$$

become

$$\frac{\sqrt{1+y^2}}{\sqrt{2}\sqrt{1+y+y^2}} < x < \frac{\sqrt{1+y^2}}{\sqrt{2}\sqrt{1-y+y^2}}$$

and, integrating with respect to x , we obtain

$$\begin{aligned} & \frac{1}{1+y^2} \left[\arcsin \left(\frac{\sqrt{1+y^2}}{\sqrt{2}\sqrt{1-y+y^2}} \right) - \arcsin \left(\frac{\sqrt{1+y^2}}{\sqrt{2}\sqrt{1+y+y^2}} \right) \right] \\ &= \frac{1}{1+y^2} \left[\arctan \left(\frac{\sqrt{1+y^2}}{|1-y|} \right) - \arctan \left(\frac{\sqrt{1+y^2}}{1+y} \right) \right] \\ &= \begin{cases} \frac{1}{1+y^2} \arctan \left(y\sqrt{1+y^2} \right) & \text{if } 0 < y < 1, \\ \frac{1}{1+y^2} \arctan \left(\frac{\sqrt{1+y^2}}{y^2} \right) & \text{if } 1 < y < \infty \end{cases} \end{aligned}$$

since

$$2(1 \pm y + y^2) - (1 + y^2) = 1 \pm 2y + y^2 = |1 \pm y|^2$$

and by the addition formula for the arctangent. Symmetry between $0 < y < 1$ and $1 < y < \infty$ allows us to focus on the former when integrating over y and to multiply the final result by two.

8. ADDENDUM

I am grateful to M. Larry Glasser for reducing $E(c)$ in Section 5 to

$$\frac{4\sqrt{2}}{3\pi} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{5}{4}, \frac{7}{4}; 1\right)$$

where ${}_3F_2$ is a hypergeometric function

$${}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{5}{4}, \frac{7}{4}; x\right) = \frac{3}{4\sqrt{2}\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)^2 \Gamma(n+3/2)}{\Gamma(n+5/4) \Gamma(n+7/4)} \frac{x^n}{n!}$$

and Michael S. Milgram [8] for pointing out the further simplification

$$\frac{\pi^4 + 8\Gamma\left(\frac{5}{8}\right)^4 \Gamma\left(\frac{7}{8}\right)^4}{2\pi^3 \Gamma\left(\frac{5}{8}\right)^2 \Gamma\left(\frac{7}{8}\right)^2}$$

which follows from equating ${}_3F_2(1/2, 1/2, 3/2; 5/4, 7/4; 1)$ with $\Omega_{1,1}(1/2, 1/2, 5/4)$ in Example 13 of [9]. Much more relevant material can be found at [10], including experimental computer runs that aided theoretical discussion here.

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